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Recurrence relations for angular momentum traces

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Abstract. It is shown that $\mathrm{Tr}(J_\lambda{}^{2p})$ can be developed from $\mathrm{Tr}(J_\lambda{}^{2p-2})$ by means of recurrence relations. Starting from $\mathrm{Tr}(J_\lambda{}^0)=\mathrm{Tr}(I)$, traces of $J_\lambda{}^{2p}$ with $p \geq 1$ are obtained. It follows as a corollary that the Bernoulli polynomials $B_n(x)$, $n \geq 2$, can be generated by means of the recurrence relations developed herein.

1. Introduction

Ambler *et al* (1962a, b) have first tabulated the traces of products of angular momentum matrices and Rose (1962) has used recoupling and graphical methods to evaluate them. Interest has once again revived in the evaluation of such traces (Witschel 1971, 1975, Subramanian and Devanathan 1974, De Meyer and Vanden Berghe 1978a,b, Rashid 1979). De Meyer and Vanden Berghe (1978b) were the first to obtain a set of recurrence relations by means of which analytical expressions for $\mathrm{Tr}(J_-^kJ_z^lJ_+^k)$ could be built up in a chain procedure. Rashid (1979) has obtained a symmetrical and computationally advantageous analytic expression for $\mathrm{Tr}(J_-^kJ_z^lJ_+^k)$. It has been proved (cf Subramanian and Devanathan 1974) that the trace of a product of angular momentum matrices given either in the spherical or the Cartesian basis can be expanded in terms of $\mathrm{Tr}(J_\lambda{}^{2p})$. In this article we develop $\mathrm{Tr}(J_\lambda{}^{2p})$ from $\mathrm{Tr}(J_\lambda{}^{2p-2})$ ($\lambda = x$ or y or z) by means of recurrence relations.

In § 2 we derive a differential recurrence relation satisfied by the trace polynomials and using this we obtain the recurrence relations between the coefficients of the trace polynomials in § 3. Results on the evaluation of $\mathrm{Tr}(J_\lambda{}^{2p})$ for $p = 6, 7, 8, 9$ are presented in § 4. It is pointed out that the Bernoulli polynomials can be generated by means of the recurrence relations obtained by us.

2. Differential recurrence relation

The present authors (Subramanian and Devanathan 1974) have shown that the trace of a product of angular momentum matrices is a polynomial in η, the eigenvalue of the J^2 operator. Recently Kaplan and Zia (1979) have rederived this result using a quasi-classical formalism.

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Thus when $p \geqslant 1$ (cf Subramanian and Devanathan 1974)

$$\mathrm{Tr}(J_\lambda^{2p}) = \Omega G_{p-1}(\eta) = [2/(2p+1)][B_{2p+1}(j+1)] \tag{2.1}$$

or, equivalently,

$$\mathrm{Tr}(J_\lambda^{2p-2}) = [2/(2p-1)][B_{2p-1}(j+1)]. \tag{2.2}$$

Here j is the angular momentum quantum number. The quantities η and Ω are given by

$$\eta = j(j+1), \qquad \Omega = \eta(2j+1). \tag{2.3}$$

$G_r(\eta)$ is a polynomial in η of degree r. $B_q(S)$ is the Bernoulli polynomial of the first kind of degree q in S and satisfies the relation (cf Miller 1960)

$$(\mathrm{d}/\mathrm{d}S)B_q(S) = qB_{q-1}(S), \qquad q \geqslant 1. \tag{2.4}$$

It may be deduced that $J_\lambda^0 = I$, the unit matrix. Using equations (2.1)–(2.4) and the relations

$$[\mathrm{d}g/\mathrm{d}(j+1)] = (2j+1)(\mathrm{d}g/\mathrm{d}\eta), \tag{2.5}$$

$$(\mathrm{d}/\mathrm{d}\eta)\{(2j+1)f(\eta)\} = [(4\eta+1)(\mathrm{d}f/\mathrm{d}\eta)+2f]/(2j+1), \tag{2.6}$$

we derive the following differential equation:

$$\begin{aligned}\eta(4\eta+1)(\mathrm{d}^2/\mathrm{d}\eta^2)G_{p-1}(\eta) + 2(7\eta+1)(\mathrm{d}/\mathrm{d}\eta)G_{p-1}(\eta) + 6G_{p-1}(\eta)\\ = [2p(2p-1)\,\mathrm{Tr}(J_\lambda^{2p-2})]/(2j+1).\end{aligned} \tag{2.7}$$

We consider the following three cases.

Case 1. $p = 1$.

From equation (2.7), we get $G_0(\eta) = \frac{1}{3}$ and hence

$$\mathrm{Tr}(J_\lambda^2) = \Omega G_0(\eta) = \Omega/3. \tag{2.8}$$

Case 2. $p = 2$.

Since $G_1(\eta) = a_1\eta + a_0$, we obtain from equations (2.7) and (2.8)

$$\mathrm{Tr}(J_\lambda^4) = \Omega G_1(\eta) = \Omega(3\eta - 1)/15. \tag{2.9}$$

Case 3. $p \geqslant 3$.

Equation (2.7) now simplifies to

$$\begin{aligned}\eta(4\eta+1)(\mathrm{d}^2/\mathrm{d}\eta^2)G_{p-1}(\eta) + 2(7\eta+1)(\mathrm{d}/\mathrm{d}\eta)G_{p-1}(\eta) + 6G_{p-1}(\eta)\\ = 2p(2p-1)\eta G_{p-2}(\eta)\end{aligned} \tag{2.10}$$

since

$$\mathrm{Tr}(J_\lambda^{2p-2}) = \Omega G_{p-2}(\eta). \tag{2.11}$$

Thus from equation (2.7) we are able to obtain our earlier expressions (Subramanian and Devanathan 1974) for $\mathrm{Tr}(J_\lambda^2)$ and $\mathrm{Tr}(J_\lambda^4)$ starting from the simplest relation

$$\mathrm{Tr}(J_\lambda^0) = \mathrm{Tr}(I) = 2j+1. \tag{2.12}$$

Equation (2.10) is the differential recurrence relation satisfied by the trace polynomials $G_{p-1}(\eta)$ and $G_{p-2}(\eta)$.

3. Recurrence relations between the coefficients of trace polynomials

Since $G_{p-1}(\eta)$ and $G_{p-2}(\eta)$ are polynomials in η of degree $(p-1)$ and $(p-2)$ respectively, let

$$G_{p-1}(\eta)=\sum_{i=0}^{p-1} a_i\eta^i, \tag{3.1}$$

$$G_{p-2}(\eta)=\sum_{i=0}^{p-2} b_i\eta^i. \tag{3.2}$$

From equations (2.10), (3.1) and (3.2) we obtain the following relations when $p\geqslant 3$ and $i=1,2,\ldots,p-2$.

(i) $$(2p+1)a_{p-1}=(2p-1)b_{p-2}, \tag{3.3}$$

(ii) $$(i+1)\{(i+2)a_{i+1}+2(2i+3)a_i\}=2p(2p-1)b_{i-1}, \tag{3.4}$$

(iii) $$a_1+3a_0=0. \tag{3.5}$$

Equations (3.3) and (3.4) are the recurrence relations between the coefficients of the trace polynomials. From them one can obtain a_{p-i} $(i=1,2,3,\ldots,p-1)$ and then, from equation (3.5), a_0 can be obtained. Thus all the coefficients of $G_{p-1}(\eta)$ can be derived from those of $G_{p-2}(\eta)$. Hence $\mathrm{Tr}(J_\lambda^{2p})$ can be generated from $\mathrm{Tr}(J_\lambda^{2p-2})$.

4. Results

Starting from

$$\mathrm{Tr}(J_\lambda^4)=\Omega(3\eta-1)/15, \tag{4.1}$$

$\mathrm{Tr}(J_\lambda^{2p})$ for $p\geqslant 3$ can be developed. We have retrieved our earlier results (Subramanian and Devanathan 1974) for $\mathrm{Tr}(J_\lambda^{2p})$ when $p=3,4,5$. For the sake of illustration, we have also obtained $\mathrm{Tr}(J_\lambda^{2p})$ for $p=6,7,8,9$. Our results are presented below:

$$\mathrm{Tr}(J_\lambda^{12})=(\Omega/1365)(105\eta^5-525\eta^4+1435\eta^3-2360\eta^2+2073\eta-691), \tag{4.2}$$

$$\mathrm{Tr}(J_\lambda^{14})=(\Omega/45)(3\eta^6-21\eta^5+84\eta^4-220\eta^3+359\eta^2-315\eta+105), \tag{4.3}$$

$$\mathrm{Tr}(J_\lambda^{16})=(\Omega/255)(15\eta^7-140\eta^6+770\eta^5-2930\eta^4+7595\eta^3-12\,370\eta^2 +10\,851\eta-3617), \tag{4.4}$$

$$\mathrm{Tr}(J_\lambda^{18})=(\Omega/1995)(105\eta^8-1260\eta^7+9114\eta^6-47\,418\eta^5+178\,227\eta^4 -460\,810\eta^3+750\,167\eta^2-658\,005\eta+219\,335). \tag{4.5}$$

The correctness of our results can be checked from the fact that

$$G_{p-1}(\eta)=\sum_{i=0}^{p-1} a_i\eta^i, \qquad p\geqslant 1, \tag{4.6}$$

satisfies the following relations.

(i) $$a_{p-1}=(2p+1)^{-1}, \tag{4.7}$$

(ii) $$a_0=2B_{2p}, \tag{4.8}$$

(iii) $$3(2^{2p-2})G_{p-1}(\tfrac{3}{4})=1, \tag{4.9}$$

(iv) $$3G_{p-1}(2)=1. \tag{4.10}$$

Here B_{2p} are the Bernoulli numbers (cf Miller 1960). Equation (4.7) follows from equations (2.8), (2.9) and (3.3). The proof of equation (4.8) is given in the Appendix. From

$$\mathrm{Tr}(J_\lambda{}^{2p})=\sum_{m=-j}^{j} m^{2p}, \tag{4.11}$$

$m=-\frac{1}{2},\frac{1}{2}$ when $j=\frac{1}{2}$ and $m=-1,0,1$ when $j=1$, equations (4.9) and (4.10) can be obtained. We have checked that our results satisfy equations (4.7)–(4.10) and also equation (2.7).

The advantage of the present method is that we are able to build up $\mathrm{Tr}(J_\lambda{}^{2p})$ from $\mathrm{Tr}(J_\lambda{}^0)=\mathrm{Tr}(I)$ by means of recurrence relations which are complete in themselves. Moreover, it can be seen from equations (2.1)–(2.6) that when $p\geqslant 1$

(i) $$B_{2p}(x)=\tfrac{1}{2}[(6u+1)G_{p-1}(u)+u(4u+1)(\mathrm{d}/\mathrm{d}u)G_{p-1}(u)], \tag{4.12}$$

(ii) $$B_{2p+1}(x)=[(2p+1)/2]\omega G_{p-1}(u), \tag{4.13}$$

where

$$u=x^2-x, \qquad \omega=u(2x-1). \tag{4.14}$$

Thus the Bernoulli polynomials $B_n(x)$, $n\geqslant 2$, can be generated, as a corollary, by means of the recurrence relations obtained in §§ 2 and 3. It may be noted that $B_{2p}(x)$ is a polynomial in u of degree p. This is consistent with the result obtained by one of us earlier (Subramanian 1974).

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Appendix

In this Appendix we prove equation (4.8). From equations (2.1) and (4.6) we have

$$a_0=\lim_{j\to 0} G_{p-1}(\eta)=2(2p+1)^{-1}\lim_{j\to 0} B_{2p+1}(j+1)/[j(j+1)(2j+1)]. \tag{A1}$$

Using equation (2.4), l'Hôpital's rule and the relations (cf Miller 1960)

$$B_{2p+1}(1)=0, \qquad p\geqslant 1, \tag{A2}$$

$$B_{2r}(1)=B_{2r}(0)=B_{2r}, \qquad r\geqslant 0, \tag{A3}$$

we obtain

$$a_0=2B_{2p}, \qquad p\geqslant 1. \tag{A4}$$

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